A guide to cluster robust inference using boottest and summclust in Stata

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Introduction

- This talk is *very* loosely based on MacKinnon, Nielsen and Webb (2021a).
- Brief overview of the cluster robust variance estimator and the wild cluster bootstrap.
- Simulation results for difficult cases.
- Overview of some diagnostic tools, especially `summclust` command.
- Quick summary of the `boottest` command.
- We focus on what Abadie, Athey, Imbens and Wooldridge (2017) calls the "model-based" approach, according to which every sample can be thought of as a random outcome, or drawing, from some meta-population.

Background on Cluster Robust Inference

Consider the following model:

$$\boldsymbol{y}_g = \boldsymbol{X}_g\boldsymbol{\beta} + \boldsymbol{u}_g, \quad g = 1, \ldots, G. \tag{1}$$

If we assume that the data are generated by (1) with $\boldsymbol{\beta} = \boldsymbol{\beta}_0$, then the OLS estimator of $\boldsymbol{\beta}$ is

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}^\top\boldsymbol{y} = \boldsymbol{\beta}_0 + (\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}^\top\boldsymbol{u}.$$

it follows that:

$$\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0 = (\boldsymbol{X}^\top\boldsymbol{X})^{-1}\sum_{g=1}^{G}\boldsymbol{X}_g^\top\boldsymbol{u}_g = \Big(\sum_{g=1}^{G}\boldsymbol{X}_g^\top\boldsymbol{X}_g\Big)^{-1}\sum_{g=1}^{G}\boldsymbol{s}_g, \tag{2}$$

where $\boldsymbol{s}_g = \boldsymbol{X}_g^\top\boldsymbol{u}_g$ denotes the $k \times 1$ score vector corresponding to the g^{th} cluster.

Variance Estimator

Dividing the sample into clusters only becomes meaningful if we further assume that

$$\mathrm{E}(\boldsymbol{s}_g \boldsymbol{s}_g^\top) = \boldsymbol{\Sigma}_g \quad \text{and} \quad \mathrm{E}(\boldsymbol{s}_g \boldsymbol{s}_{g'}^\top) = \boldsymbol{0}, \quad g, g' = 1, \ldots, G, \quad g' \neq g. \tag{3}$$

An estimator of the variance of $\hat{\beta}$ should be based on the usual sandwich formula,

$$(\boldsymbol{X}^\top \boldsymbol{X})^{-1} \Big(\sum_{g=1}^{G} \boldsymbol{\Sigma}_g \Big) (\boldsymbol{X}^\top \boldsymbol{X})^{-1}. \tag{4}$$

The natural way to estimate (4) is to replace the $\boldsymbol{\Sigma}_g$ matrices by their empirical counterparts, which yields the cluster-robust variance estimator, or CRVE,

$$\mathrm{CV}_1: \qquad \frac{G(N-1)}{(G-1)(N-k)} (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \Big(\sum_{g=1}^{G} \hat{\boldsymbol{s}}_g \hat{\boldsymbol{s}}_g^\top \Big) (\boldsymbol{X}^\top \boldsymbol{X})^{-1}. \tag{5}$$

What Can Go Wrong

- The CRVE can work well, but the asymptotics depend on G, the number of clusters.
- The CRVE can work poorly when there are few clusters.
- The CRVE also runs into problems when the clusters are heterogeneous:
  - differing size clusters.
  - Unequal distribution of X; cluster specific treatment is an extreme example.
- The wild cluster bootstrap (Cameron, Gelbach and Miller, 2008; Djogbenou, MacKinnon and Nielsen, 2019) often, but not always, works better than the CRVE.

The Wild Cluster Bootstrap

The restricted version of the wild cluster bootstrap (WCR) works as follows:

- Suppose that $\tilde{\beta}$ denotes the OLS estimate of β subject to the restriction $\boldsymbol{a}^\top\beta = \boldsymbol{a}^\top\beta_0$. Then $\tilde{\boldsymbol{u}}_g = \boldsymbol{y}_g - \boldsymbol{X}_g\tilde{\beta}$ denotes the vector of restricted residuals for the g^{th} cluster. The Bootstrap DGP is

$$\boldsymbol{y}_g^{*b} = \boldsymbol{X}_g\tilde{\beta} + \boldsymbol{u}_g^{*b}, \quad \boldsymbol{u}_g^{*b} = v_g^{*b}\tilde{\boldsymbol{u}}_g, \quad g = 1, \ldots, G, \tag{6}$$

  where the v_g^{*b} are independent realizations of an auxiliary random variable v^* .
- Typically, the best choice for v^* is the Rademacher distribution, in which case v^* equals 1 or -1 with equal probabilities Davidson and Flachaire (2008), Djogbenou et al. (2019).
- Then B bootstrap samples are generated, the full model is estimated with the bootstrap samples and either a bootstrap P value or C.I. is calculated.

Figure: Rejection frequencies as G changes, $\gamma = 3$, $\rho = 0.10$

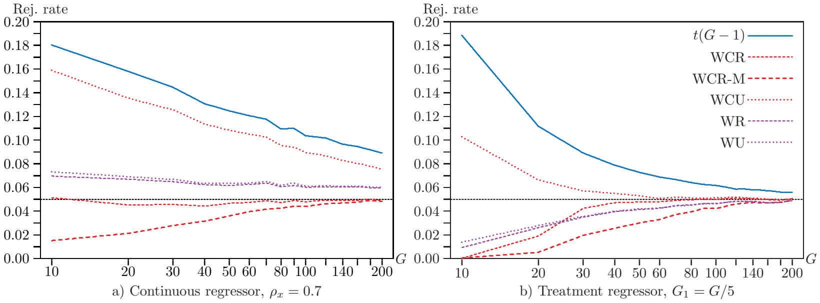


Figure from previous working paper version of Djogbenou, MacKinnon and Nielsen (2019)

Figure: Rejection frequencies for continuous regressor, $G = 20$, $N = 4000$, $\rho = 0.10$

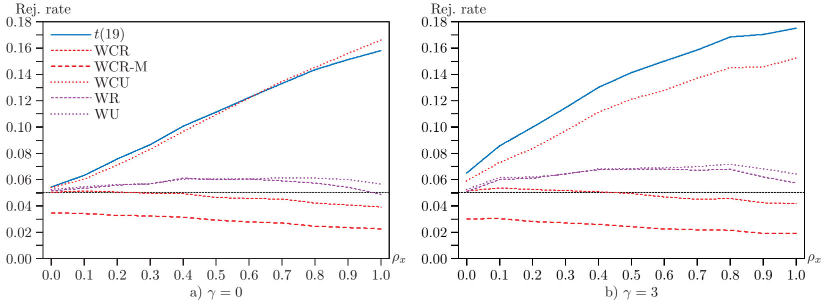


Figure from previous working paper version of Djogbenou, MacKinnon and Nielsen (2019)

Figure: Rejection frequencies for treatment dummy, $G = 20$, $N = 4000$, $\rho = 0.10$

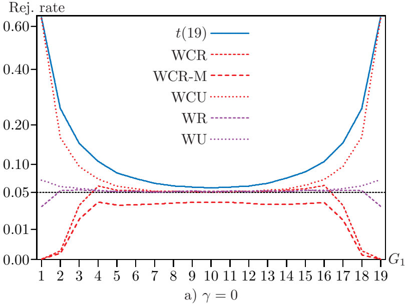


Figure from previous working paper version of Djogbenou, MacKinnon and Nielsen (2019)

When Will CRVEs be Unreliable?

- There are a few diagnostics one can examine to check whether CV_1 is likely to be reliable.
- Carter, Schnepel and Steigerwald (2017) propose the effective number of clusters, G^*.
- This can be calculated using the Stata package `clusteff` described in Lee and Steigerwald (2018).
- The forthcoming `summclust` package calculates G^* more efficiently.
- When G^* differs significantly from G then inference based on $t \sim t(G-1)$ is likely to be unreliable.
- In those situations you can alternatively use WCR or $t \sim t(G^*-1)$; see MacKinnon and Webb (2017) for details.
- The following directly will host summclust on github shortly.

Cluster Level Leverage

MacKinnon, Nielsen and Webb (2021b) proposes a cluster level measure of leverage.

If we drop the g^{th} cluster when we estimate $\boldsymbol{\beta}$, the g^{th} residual vector changes from $\hat{\boldsymbol{u}}_g$ to $(\mathbf{I} - \boldsymbol{H}_g)^{-1}\hat{\boldsymbol{u}}_g$, where

$$\boldsymbol{H}_g = \boldsymbol{X}_g(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}_g^\top \tag{7}$$

is the $N_g \times N_g$ diagonal block of the hat matrix that corresponds to cluster g.

As a measure of leverage, we can instead use a matrix norm of the $\boldsymbol{H}_g$.

$$L_g = \mathrm{Tr}(\boldsymbol{H}_g) = \mathrm{Tr}\big(\boldsymbol{X}_g^\top\boldsymbol{X}_g(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\big). \tag{8}$$

Partial Leverage

The partial leverage of observation i is simply the i^{th} diagonal element of the matrix $\acute{\boldsymbol{x}}_j(\acute{\boldsymbol{x}}_j^\top\acute{\boldsymbol{x}}_j)^{-1}\acute{\boldsymbol{x}}_j^\top$, which is just $\acute{x}_{ji}^2/(\acute{\boldsymbol{x}}_j^\top\acute{\boldsymbol{x}}_j)$.
The analogous measure of partial leverage for cluster g is

$$L_{gj} = \frac{\acute{\boldsymbol{x}}_{gj}^\top\acute{\boldsymbol{x}}_{gj}}{\acute{\boldsymbol{x}}_j^\top\acute{\boldsymbol{x}}_j}, \tag{9}$$

where $\acute{\boldsymbol{x}}_{gj}$ is the subvector of $\acute{\boldsymbol{x}}_j$ corresponding to the g^{th} cluster.
The average of the L_{gj} is evidently $1/G$, so that if cluster h has $L_{hj} >> 1/G$, it has high partial leverage for the j^{th} coefficient.

Cluster Level Influence

MacKinnon et al. (2021b) also proposes a cluster level measure of influence. As an example, consider using a regression to estimate a sample mean. We can rewrite the expression for $\hat{\beta}$ as

$$\hat{\beta} = \sum_{g=1}^{G} \frac{N_g}{N} \bar{y}_g = \sum_{g=1}^{G} L_g \hat{\beta}_g, \tag{10}$$

so that $\hat{\beta}$ is seen to be a weighted average of the G estimates $\hat{\beta}_g = \bar{y}_g$, with the weight for each cluster equal to its leverage. Similarly, we find that

$$\hat{\beta}^{(g)} = \frac{N}{N - N_g} \sum_{h \neq g} L_h \hat{\beta}_h, \tag{11}$$

Subtracting (10) from (11), we conclude that

$$\hat{\beta}^{(g)} - \hat{\beta} = L_g \big(\hat{\beta}^{(g)} - \hat{\beta}_g\big) = \frac{N_g}{N} \big(\hat{\beta}^{(g)} - \hat{\beta}_g\big). \tag{12}$$

Therefore, cluster g will be influential whenever omitting it yields an estimate $\hat{\beta}^{(g)}$ that differs substantially from the estimate $\hat{\beta}_g$ for cluster g itself, especially when cluster g also has high leverage.

Alternatives to bootstrapping

- While the wild cluster bootstrap works well it can sometimes fail.
- Alternative CRVEs are sometimes reliable but computationally infeasible with large clusters :

$$\text{CV}_2: \qquad (\boldsymbol{X}^\top \boldsymbol{X})^{-1}\Big(\sum_{g=1}^{G} \ddot{\boldsymbol{s}}_g \ddot{\boldsymbol{s}}_g^\top\Big)(\boldsymbol{X}^\top \boldsymbol{X})^{-1}. \tag{13}$$

  In the middle factor here,

$$\ddot{\boldsymbol{s}}_g = \boldsymbol{X}_g^\top \boldsymbol{M}_g^{-1/2} \hat{\boldsymbol{u}}_g, \quad \text{where} \quad \boldsymbol{M}_g = \mathbf{I}_{N_g} - \boldsymbol{X}_g(\boldsymbol{X}^\top \boldsymbol{X})^{-1}\boldsymbol{X}_g^\top. \tag{14}$$

  - In Stata, see the `reg_sandwich` package.
  - Tyszler, M., Pustejovsky, J. E., & Tipton, E. 2017.
- See also Randomization Inference and other forms of randomization MacKinnon and Webb (2020), Cai, Canay, Kim and Shaikh (2021), Canay, Romano and Shaikh (2017) and references therein.
  - In Stata, see the `RITEST` package, by Simon Hess.

Multi-way Clustering

- Clustering can occur in more than one dimension.
- Cameron et al. (2011) proposed a variance estimator of $\hat{\beta}$

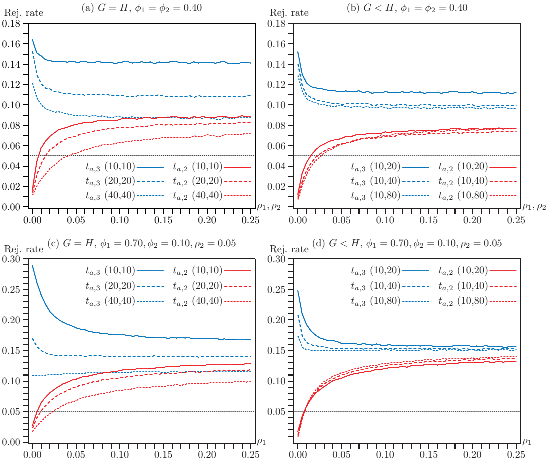
$$\widehat{\text{Var}}(\hat{\beta}) = (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \hat{\Sigma} (\boldsymbol{X}^\top \boldsymbol{X})^{-1}$$

$$\hat{\boldsymbol{\Sigma}} = \sum_{g=1}^{G} \hat{\boldsymbol{s}}_g \hat{\boldsymbol{s}}_g^\top + \sum_{h=1}^{H} \hat{\boldsymbol{s}}_h \hat{\boldsymbol{s}}_h^\top - \sum_{g=1}^{G} \sum_{h=1}^{H} \hat{\boldsymbol{s}}_{gh} \hat{\boldsymbol{s}}_{gh}^\top.$$

- MacKinnon, Nielsen and Webb (2021c) proposes a multi-way cluster bootstrap.
- Multi-way theory is still under active development (Chiang, Kato and Sasaki, 2020; Chiang, Kato, Ma and Sasaki, 2021; Davezies, D'Haultfœuille and Guyonvarch, 2021; Menzel, 2021).

Figure from MacKinnon et al. (2021c)

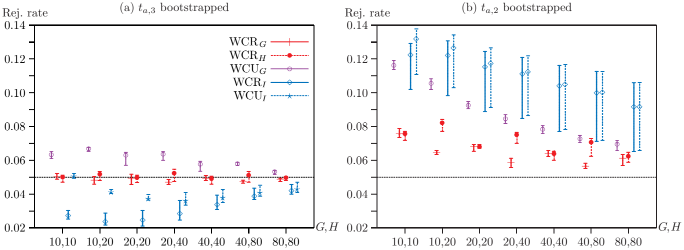
Figure: Rejection frequencies for two-way t-tests



Notes: There are 400,000 replications, and the sample size N is always 6400. All tests are at the 5% nominal level.

Figure from MacKinnon et al. (2021c)

Figure: Rejection frequencies for wild cluster bootstrap tests



Notes: There are 100,000 replications, and $N = 6400$. All bootstrap tests use $B = 399$ and reject whenever $\hat{P}^*_S < 0.05$. In all cases, $\phi_1 = \phi_2 = 0.40$. For each method and each pair of G, H values, the top of the vertical line shows the largest observed rejection frequency across the cases $\rho_1 = \rho_2 = 0.01, 0.02, \ldots, 0.10$, the bottom of the line shows the smallest one, and the mean over the ten frequencies is shown by a symbol.

boottest

- In Stata, the program `boottest` handles many of these routines.
- Roodman et al. (2019) describes the features of the program and how it achieves computational efficiency.
- `boottest` itself is for estimating bootstrap P values and confidence intervals.
- `waldtest` is contained within boottest and can be used for asymptotic P values and confidence intervals.

waldtest

$$y_i = \alpha + \beta x_i + \gamma w_i + \epsilon_i$$

- Imagine you are interested in estimating the above model.
- You want to test the null hypothesis $H_0 : \beta_0 = 0$ under different assumptions about the level of clustering: city, state, etc .
- It can also handle multi-way clustering, such as state and year.

Example

```
reg y x w, robust
waldtest x, cluster(city)
waldtest x, cluster(state)
waldtest x, cluster(state year)
```

boottest

$$y_i = \alpha + \beta x_i + \gamma w_i + \epsilon_i$$

- Consider the same set up as before, but now you wish to use a bootstrap procedure to test the null hypothesis $H_0 : \beta_0 = 0$.
- The following example shows how to do so for: the wild cluster bootstrap WCR (clustering by state); the wild bootstrap WR clustering by state (MacKinnon and Webb, 2018); and multi-way clustered by state and year.

Example

```
reg y x w, robust
boottest x, cluster(state)
boottest x, cluster(state) bootcluster(obsid)
boottest x, cluster(state year) bootcluster(year)
```

Some Guidance

- For each plausible level of clustering examine the distribution of cluster sizes.
- Settle on a level of clustering, perhaps by testing .
- For key regressions report measures of cluster level influence, leverage, and the effective number of clusters, shortly available with `summclust`.
- Employ the wild cluster bootstrap by default, easily done with `boottest`.
- Consider alternative means of inference with few treated clusters.

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