A generalized boxplot for skewed and heavy-tailed distributions implemented in Stata

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UK Stata users meeting

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Roadmap

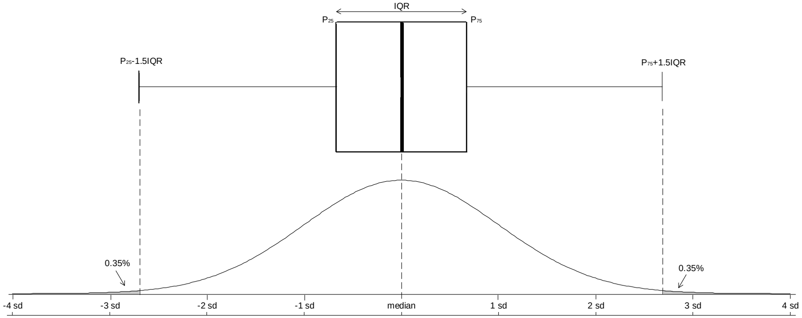
Structure of the presentation

- Introduction
- Preamble (Tukey g and h: $T_{g,h}$)
- A generalized boxplot
- Simulations
- Examples (Eartquakes in Latin America and Footballers' wages)
- Stata command
- Conclusion
- References

Univariate outliers identification

Standard Boxplot, Standard Normal distribution

- X is the $(n \times 1)$ data vector (n individuals, 1 variable)



Univariate outliers identification

Standard Boxplot, heavy tailed t_2 distribution

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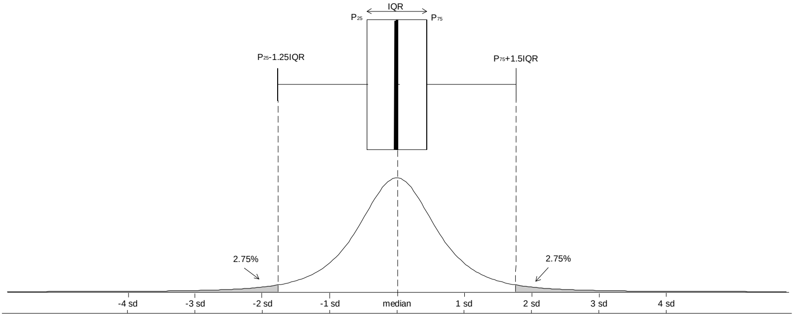

Univariate outliers identification

Standard Boxplot, skewed χ^2_5 distribution

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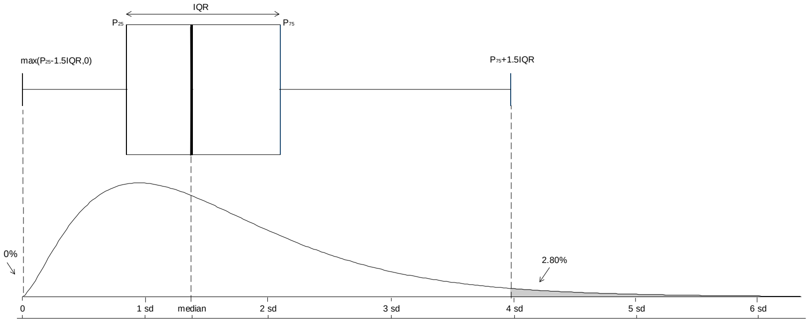


Univariate outliers identification

Limitations of the boxplot

- Only suited for (almost) symmetric data and (approximately) mesokurtic distributions

Solution 1

Modify the whiskers of the boxplot to deal with asymmetry

- **Adjusted Boxplot** (Hubert and Vandervieren, 2008).
  - The whiskers of the boxplot are moved according to a robust measure of asymmetry, the medcouple ($-1 \leq MC \leq 1$):

$$\begin{cases} \left[Q_{0.25} - 1.5e^{-4\,\mathrm{MC}}\,\mathrm{IQR}\,;\,Q_{0.75} + 1.5e^{3\,\mathrm{MC}}\,\mathrm{IQR}\right] & \text{if MC} \geq 0 \\ \left[Q_{0.25} - 1.5e^{-3\,\mathrm{MC}}\,\mathrm{IQR}\,;\,Q_{0.75} + 1.5e^{4\,\mathrm{MC}}\,\mathrm{IQR}\right] & \text{if MC} < 0, \end{cases}$$

  - Copes well with asymmetry ($MC \leq 0.6$) but does not take (explicitly) into accout heavyness of tails
  - Rejection rate set to 0.7%
  - Rule based on simulations
  - Computational complexity $O(n \log n)$ (see Gelade et *al.*, 2014).

Univariate outliers identification

Limitations of the boxplot

- Only suited for (almost) symmetric data and (approximately) mesokurtic distributions

Solution 2

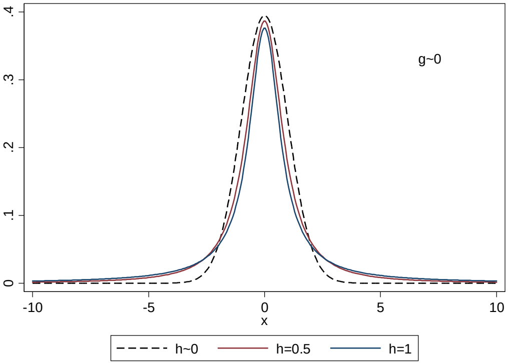
Modify the whiskers of the boxplot to deal with asymmetry and tail heavyness

- **Generalized Boxplot**
  - Do a rank preserving transformation of the data to end-up with a known distribution
  - Use the theoretical quantiles of the latter to set whiskers (after applying an inverse transformation)
  - Cope with both the skewness and tail heavyness
  - Set the desired rejection rate to any chosen level
  - Computational complexity $O(n)$ (as the standard boxplot)

Heavy-tailed distributions

Definition

If $Z \sim N(0,1)$, $g \neq 0$ and $h \in \mathbb{R}$, the random variable Y is said to be $T_{g,h}$ distributed if $Y = \frac{1}{g}\left[\exp\left(gZ\right) - 1\right] \exp\left(hZ^2/2\right)$

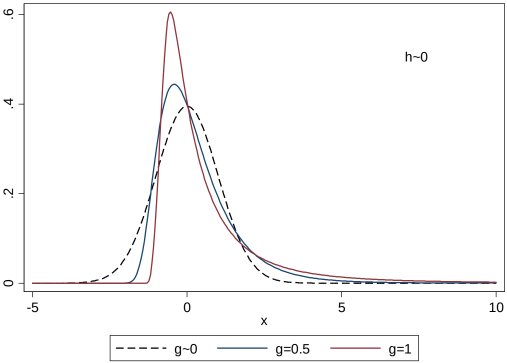


Preamble: Tukey g and h distribution

Asymmetrical distributions

Definition

If $Z \sim N(0,1)$, $g \neq 0$ and $h \in \mathbb{R}$, the random variable Y is said to be $T_{g,h}$ distributed if $Y = \frac{1}{g}\left[\exp\left(gZ\right)-1\right]\exp\left(hZ^2/2\right)$



Preamble: Tukey g and h distribution

Asymmetrical and heavy-tailed distributions

Definition

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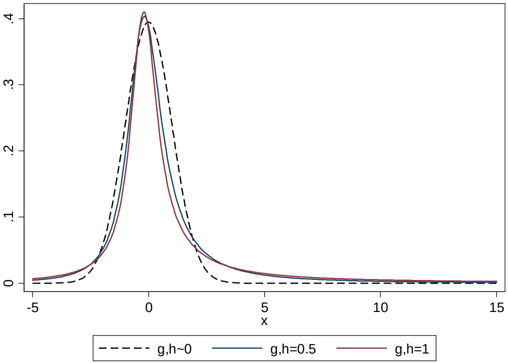

Univariate outliers identification

Standard Boxplot

- An outlier is defined as any observation lying outside the fence defined by whiskers $P_{25} - 1.5\,\mathrm{IQR}$ and $P_{75} + 1.5\,\mathrm{IQR}$

Theoretical detection rate α

- More generally, a theoretical detection rate equal to α is given by $[Q_{0.25} - c(\alpha)\,\mathrm{IQR}\,;\, Q_{0.75} + c(\alpha)\,\mathrm{IQR}]$ with $c(\alpha) = \frac{z_{1-\alpha/2} - z_{0.75}}{z_{0.75} - z_{0.25}}$ where z_p denotes the quantile of order p of the standard normal distribution.

Limitations of the boxplot

- Only suited for (almost) symmetric data and (approximately) mesokurtic distributions

Solution

- Modify the boxplot to deal with asymmetry and tail heavyness.

Procedure

Transformation

For an initial dataset $\{x_1, \ldots, x_n\}$, the guidelines of the new method are the following:

1. Center and reduce the data: $x_i^* = \frac{x_i - m^0}{s_0}$ where $s_0 = \mathrm{IQR}(\{x_j\})$ and $m_0 = \mathrm{Q}_{0.5}(\{x_j\})$

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5. Center and reduce the values w_i: $w_i^* = \frac{w_i - Q_{0.5}(\{w_j\})}{\mathrm{IQR}(\{w_j\})/1.3426}$

Transformation

6. Adjust the distribution of the values w_i^* $(i = 1, \ldots, n)$ by the Tukey $T_{\widehat{g}^*,\widehat{h}^*}$ distribution:

$$\widehat{g} = \frac{1}{z_{0.9}} \ln\left(-\frac{P_{0.9}\left(\{w_j^*\}\right)}{P_{0.1}\left(\{w_j^*\}\right)}\right), \qquad \widehat{h} = \frac{2\ln\left(-\widehat{g}\,\frac{P_{0.9}\left(\{w_j^*\}\right)P_{0.1}\left(\{w_j^*\}\right)}{P_{0.9}\left(\{w_j^*\}\right)+P_{0.1}\left(\{w_j^*\}\right)}\right)}{z_{0.9}^2}$$

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7. Select the rejection bounds (L_-^*, L_+^*) using specific quantiles of the adjusted distribution (here $P_{0.35}$ and $P_{99.65}$)

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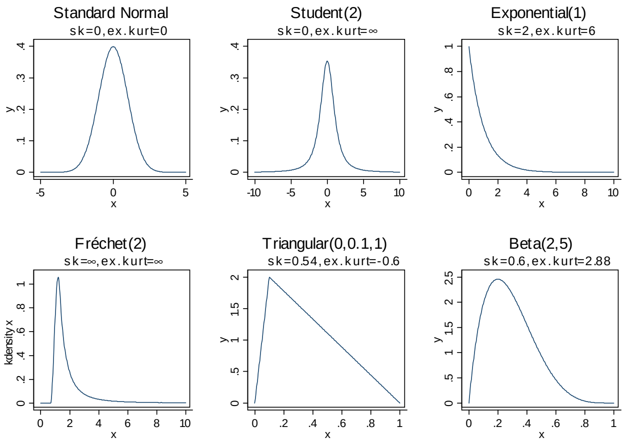
7. Select the rejection bounds (L_-^*, L_+^*) using specific quantiles of the adjusted distribution (here $P_{0.35}$ and $P_{99.65}$)
8. Build the detection bounds B_-^* and B_+^* (whiskers) for the original dataset applying the complete inverse transformation

$$f(L_\pm^*) = \Phi\left(Q_{0.5}\left(\{w_j\}\right) + \frac{\mathrm{IQR}(\{w_j\})}{1.3426} L_\pm^*\right)$$

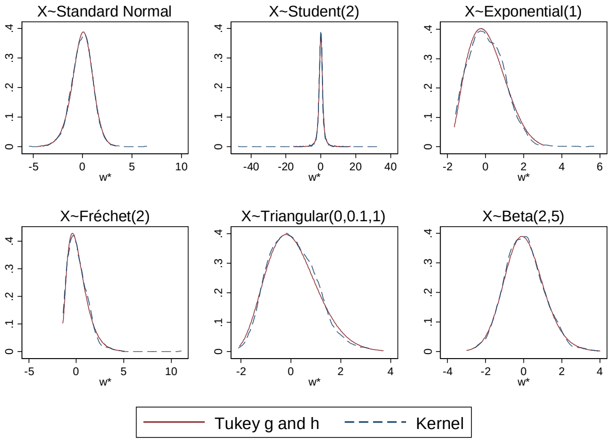
$$B_\pm^* = \left(f(L_\pm^*)\left[\min(\{r_j\}) + \max(\{r_j\})\right] + \min(\{x_j^*\}) - 0.1\right) s_0 + m_0$$

Considered distributions



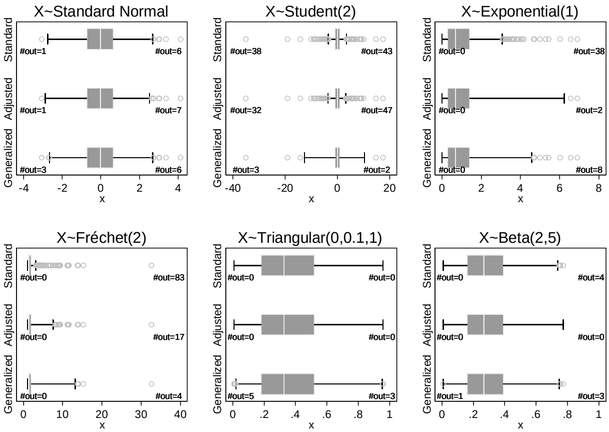
Numerical example

Quality of fit of transformed variable



Boxplot

Standard, Adjusted and Generalized boxplots



Outliers∼U(4.9,5.1) on the scale of the Normal (1000 replications)

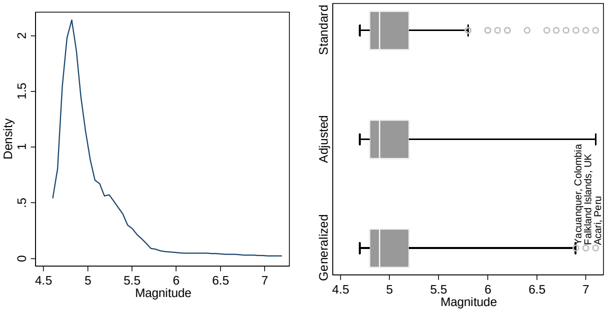
			Outliers: U(4.9,5.1)			
			Sensitivity		Specificity	
		ε	n=100	n=1000	n=100	n=1000
N(0,1)	Standard Boxplot	1%	100.00%	100.00%	99.06%	99.32%
		5%	100.00%	100.00%	99.19%	99.58%
	Adjusted Boxplot	1%	98.10%	100.00%	97.81%	99.12%
		5%	92.40%	100.00%	97.71%	98.95%
	Generalized Boxplot	1%	100.00%	100.00%	96.82%	98.91%
		5%	98.30%	100.00%	97.95%	99.45%
t2	Standard Boxplot	1%	100.00%	100.00%	91.96%	91.98%
		5%	100.00%	100.00%	92.95%	92.83%
	Adjusted Boxplot	1%	100.00%	100.00%	90.93%	91.70%
		5%	100.00%	100.00%	91.05%	91.54%
	Generalized Boxplot	1%	100.00%	100.00%	96.68%	98.47%
		5%	100.00%	100.00%	97.71%	99.15%
Exp(1)	Standard Boxplot	1%	100.00%	100.00%	94.93%	95.47%
		5%	100.00%	100.00%	96.29%	96.72%
	Adjusted Boxplot	1%	99.70%	100.00%	98.48%	99.47%
		5%	98.80%	100.00%	98.54%	99.90%
	Generalized Boxplot	1%	100.00%	100.00%	96.55%	99.35%
		5%	99.50%	100.00%	98.42%	99.95%

Sensitivity and Specificity

Outliers$\sim$U(4.9,5.1) on the scale of the Normal (1000 replications)

			Outliers: U(4.9,5.1)			
			Sensitivity		Specificity	
		ε	n=100	n=1000	n=100	n=1000
Fréchet(2)	Standard Boxplot	1%	✓ 100.00%	✓ 100.00%	✗ 91.96%	✗ 92.00%
		5%	✓ 100.00%	✓ 100.00%	✗ 93.31%	✗ 93.41%
	Adjusted Boxplot	1%	✓ 100.00%	✓ 100.00%	✗ 94.18%	! 95.26%
		5%	✓ 100.00%	✓ 100.00%	✗ 93.38%	✗ 94.54%
	Generalized Boxplot	1%	✓ 100.00%	✓ 100.00%	! 96.74%	! 98.96%
		5%	✓ 100.00%	✓ 100.00%	! 98.08%	✓ 99.57%
Triangular(0,0.1,1)	Standard Boxplot	1%	✓ 100.00%	✓ 100.00%	✓ 99.75%	✓ 99.83%
		5%	✓ 99.50%	✓ 100.00%	✓ 99.90%	✓ 99.95%
	Adjusted Boxplot	1%	✗ 53.20%	✗ 56.40%	✓ 99.39%	✓ 99.99%
		5%	✗ 34.40%	✗ 8.20%	✓ 99.23%	✓ 100.00%
	Generalized Boxplot	1%	! 98.90%	✓ 99.90%	! 96.71%	✓ 99.26%
		5%	✗ 93.80%	! 97.70%	! 97.67%	✓ 99.86%
Beta(2,5)	Standard Boxplot	1%	✓ 100.00%	✓ 100.00%	! 98.81%	✓ 99.42%
		5%	✓ 100.00%	✓ 100.00%	✓ 99.15%	✓ 99.72%
	Adjusted Boxplot	1%	✗ 76.40%	! 98.70%	✓ 99.07%	✓ 99.97%
		5%	✗ 54.60%	✗ 71.10%	! 98.62%	✓ 99.94%
	Generalized Boxplot	1%	✓ 99.60%	✓ 100.00%	! 97.26%	✓ 99.36%
		5%	! 98.48%	✓ 99.60%	! 98.48%	✓ 99.79%

Estimated medcouple: 0.43



Stata command

Syntax

`box_out` *varname* `[if] [in] [,out(`*varname*`) bdp(#) perc(#) nograph]`

Options

- *out:* Identifies the new variable to be created to identify individuals outside the fence defined by the whiskers
- *bdp:* Sets the desired Break-down point (in %). It is 10% by default
- *perc:* Sets the desired percentage of points outside the whiskers in case of uncontaminated data. It is set to 0.7% by default
- *nograph:* Suppresses the graph

Saved results and output

- *e(g), e(h):* Estimated skewness and elongation parameters of the underlying Tukey g and h distribution
- *e(lowerW), e(upperW):* Value of the lower and upper whiskers
- A basic boxplot is created but we recommend to refer to N. J. Cox, S.J. (2009) for better output

Conclusion

Generalized boxplot

We propose a very simple generalized boxplot that

- is suited for skewed and/or heavy-tailed distributions
- allows for setting the desired detection rate of atypical observation
- has a computational complexity of $O(n)$

In Stata

We provide a simple command that

- estimates the whiskers of the generalized boxplot
- creates a simple boxplot.
- we however refer to Cox (2009) and Cox(2013) for more complete graphs.

Complementary results

In multivariate analysis we have a projection based estimator

- to create a bagplot in 2D
- identify outliers for multivariate skewed and heavy-tailed distributions

References

- Bruffaerts, C., Verardi, V. and Vermandele, C., (2014). "A generalized boxplot for skewed and heavy-tailed distributions". *Statistics and Probability Letters* (forthcoming)
- Cox, N.J. (2013)."Speaking Stata: Creating and varying box plots: Correction". *Stata Journal* 13(2). pp. 398-400.
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- Gelade, W., Verardi, V. and Vermandele, C. (2014). "Time efficient algorithms for robust estimators of location, scale, symmetry and tail heaviness". *Stata Journal* (forthcoming).
- Hubert, M. and Vendervieren, E. (2008). "An adjusted boxplot for skewed distributions". *Computational Statistics & Data Analysis* 52(12). pp 5186-5201